Journal of Statistical Physics, Vol. 33, No. 1, 1983

Diffusion in a Bistable Potential: A Comparative Study of Different Methods of Solution

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Received March 29, 1983

The problem of diffusion in a bistable potential is studied by considering the associated nonlinear Langevin equation and its equivalent Fokker–Planck equation. Two numerically exact methods of solution, namely, the Monte Carlo solution of the nonlinear Langevin equation and the solution of the Fokker–Planck equation via the finite difference technique, are considered. The latter method has the advantage that it directly gives the evolution of the probability distribution function. Approximate analyses of the fluctuations using the system size expansion, the Gaussian decoupling procedure, and the scaling approach are also carried out. These investigations are performed on a representative problem for two specific cases: (1) evolution from intrinsically unstable states and (2) evolution from extensive regime. The fluctuations obtained using these approximate methods are compared with those obtained via the numerically exact methods. The study brings out the advantages and limitations of each of the methods considered.

KEY WORDS: Diffusion in bistable potential; numerical solution of the nonlinear F.P. Equation; scaling theory; generalized statistical linearization; system size expansion; Monte Carlo solution of the nonlinear Langevin equation.

1. INTRODUCTION

Fluctuations in nonlinear systems have been receiving considerable attention in recent years. In the "equation of motion" approach the analysis is carried out by considering a nonlinear Langevin equation or its equivalent Fokker–Planck equation. Exact closed-form solutions for these equations

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0022-4715/83/1000-0181$03.00/0

are rarely possible.[1] However, exact numerical integration of these equations is fairly straightforward. For example, the Monte Carlo solution of the nonlinear Langevin equation can be employed for extracting the moments of the stochastic process. Alternatively, the exact numerical solution of the Fokker–Planck equation can be obtained via the finite difference scheme. The latter method has the added advantage of yielding the probability distribution function.

On the other hand, approximate methods exist, some of which are analytical in nature[2–5] and others have numerical bias.[6–9] It is obvious that each of these would have its own advantages and limitations. Analytical methods permit insight into the nature of fluctuations, but by their very nature are limited to simple situations only (e.g., single-variable case). On the other hand, the methods with numerical bias can be used even in complicated situations (e.g., higher dimension, arbitrary nonlinearity, etc.). However, the nature of approximations is such that these methods are not uniformly applicable. Thus we find a need for a comparative study of the different methods of analysis of fluctuations in nonlinear systems. The hope of this study is that it would enable us to identify some approximate methods which can be employed in the analysis of complicated situations.

The comparative study is carried out with respect to a particular model, namely, diffusion in a bistable potential.[10] This problem has been an object of much investigation.[2,11] The diffusion can be either from an apparently unstable state (extensive regime) to a stable steady state, or from an intrinsically unstable state to two stable steady states. In the first case, where a single steady state is preferred, the system size expansion[5] and the recent linearization schemes[7–9] are expected to give reasonably good results.

The evolution from an intrinsically unstable state is a difficult problem[2–4,11–15] and has been studied in detail. The system size expansion is not applicable in this case.[3] But, the Gaussian decoupling scheme[8] can be used if one takes care of the fact that two steady states are allowed.

The approximate methods considered for intercomparison are (a) the scaling theory,[2–4] (b) system size expansion,[5] and (c) the generalized statistical linearization scheme.[7–9] The Monte Carlo solution of the nonlinear Langevin equation and the numerical solution of the Fokker–Planck equation are also considered. The Monte Carlo solution is taken as the reference against which other methods are compared.

[3] It is possible to employ system size expansion for this case also using Dekker's method.[14] He splits the problem into an irreducible part (which essentially preserves the multimodal character of the distribution function) and a corrective remainder.

The paper is organized as follows. In Section 2, we describe the model representing the diffusion in a bistable potential, the corresponding nonlinear Langevin equation, and its equivalent Fokker–Planck equation. In Section 3 we describe various methods of solutions considered. Section 4 discusses the relative performance of these methods. The principal conclusions are summarized in Section 5.

2. A MODEL FOR DIFFUSION IN A BISTABLE POTENTIAL

One of the models for diffusion in a bistable potential is represented by the nonlinear Langevin equation (NLE)

$$\frac{d}{dt}x = \gamma x - gx^3 + \eta(t) \tag{1}$$

where $x(t)$ is the driven random variable, γ is a positive friction coefficient, and g is the parameter of nonlinearity (taken to be positive for the global stability of the solution). The driving random force $\eta(t)$ is assumed to be a Gaussian white noise obeying

$$\langle \eta(t) \rangle = 0 \quad \text{and} \quad \langle \eta(t)\eta(t') \rangle = 2\epsilon\delta(t - t') \tag{2}$$

where ϵ is the diffusion coefficient. The equivalent Fokker–Planck equation (FPE) for the distribution function $P(x,t)$ is

$$\frac{\partial}{\partial t}P(x,t) = -\frac{\partial}{\partial x}\left[(\gamma x - gx^3)P(x,t)\right] + \epsilon\frac{\partial^2}{\partial x^2}P(x,t) \tag{3}$$

The potential considered permits three extrema for x $(= x_e)$ out of which $x_e = 0$ corresponds to an unstable steady state and $x_e = \pm(\gamma/g)^{1/2}$ correspond to the stable steady states.

The evolution of $x(t)$ [and hence of $P(x,t)$] depends crucially on the choice of initial conditions.[2] Let the initial distribution function be a Gaussian characterized by the mean and the variance of δ and σ_0^2, respectively. The initial variance σ_0^2 is expected to be less than or of the order of ϵ in most physical situations. We will be studying the nature of fluctuations as a function of δ. Typically two types of fluctuations arise (as a function of δ). The case $\delta^2 \gg \epsilon$ is referred to as the extensive regime.[2] This corresponds to passage from an apparently unstable state to a *preferred* stable state. Subsequent slow time evolution due to Kramers diffusion[10] will not be considered here. The more interesting case corresponds to $\delta^2 \ll \epsilon$, which is referred to as the intrinsically unstable region. This corresponds to evolution from an intrinsically unstable state to the two steady states.

3. DIFFERENT METHODS OF SOLUTION

Of the several methods of solution studied here, the first two are exact numerical methods, the next two are Gaussian representations of the actual distribution function of $x(t)$, and the last one is the scaling theory.

3.1. Monte Carlo Solution

A Monte Carlo solution of the nonlinear Langevin equation [see Eq. (1)] is carried out as follows. A random walk is started with its initial position $x(t_0)$ sampled from a given initial distribution. Then Eq. (1) is numerically integrated using the standard Runge–Kutta Gill algorithm over a time interval Δt. The random noise $\eta(t)$ is modeled to be rectangular pulses of arbitrary heights, but constant in a time interval $\tau \geqslant \Delta t$. The heights are sampled from a Gaussian distribution function with a variance of $2\epsilon\tau$ so that the normalization condition given by Eq. (2) is satisfied in the sense of an integral. The process is continued up to the desired final time t, thus completing one random walk. A large number of such random walks are generated and the required statistics are obtained by averaging over the ensemble thus produced.

It should be clear from the preceding paragraph that the method is general. Stochastic processes involving more than one variable (higher dimension) could be easily handled. Whereas most of the other methods are restricted to the driving random force being a Gaussian white noise, the Monte Carlo solution does not have such limitations. $\eta(t)$ could be sampled from any arbitrary distribution function desired and the entire analysis goes through.

3.2. Numerical Solution of the Fokker–Planck Equation

The numerical solution of the Fokker–Planck equation is obtained using finite difference techniques.[16] Natural boundary conditions $P(x,t) \to 0$ for $x \to \pm\infty$ are employed. A mesh structure is imposed on the variables x and t. The mesh widths Δx and Δt of the variables x and t, respectively, need to be chosen carefully, since the finite-difference approximations are valid only in the limit of small Δx and Δt.

Let j and k denote the indices of the x mesh and t mesh. The finite-difference equations[16] set up are

$$aP_{j,k+1} + bP_{j-1,k+1} + cP_{j+1,k+1} - P_{j,k} = 0, \qquad j = 2, \ldots, (N-1) \quad (4)$$

Here k denotes the earlier time step and N represents the number of meshes

in the x domain. The coefficients a, b, and c are given by

$$a = \left(-\gamma + 3gx_j^2 - \frac{2\epsilon}{\Delta x^2} \right) \Delta t - 1$$

$$b = \left(\frac{2\epsilon}{\Delta x^2} + \frac{\gamma x_j - g x_j^3}{2\Delta x} \right) \Delta t \tag{5}$$

$$c = \left(\frac{2\epsilon}{\Delta x^2} - \frac{\gamma x_j - g x_j^3}{2\Delta x} \right) \Delta t$$

Given an initial distribution function at time $t = 0$, these equations reduce to a tridiagonal set of equations involving $P_{j,k+1}$, $P_{j+1,k+1}$, and $P_{j-1,k+1}$. These equations are solved to obtain $P_{j,k+1}$ for $j = 2, \ldots, (N - 1)$. Now the entire solution $P(x, t)$ is obtained in time steps of Δt.

The choice of Δx is dictated by the accuracy with which $P(x, t)$ is computed at any time t. It is essential that $P(x, t)$ does not vary much over the mesh width, since the finite-difference approximation implies a constant interpolation of $P(x, t)$ in the interval Δx. This in turn implies a small Δx, since the procedure becomes numerically unstable otherwise.

For the problem considered, $\langle x(t) \rangle$ ranges typically from $-(\gamma/g)^{1/2}$ to $(\gamma/g)^{1/2}$ with the variance in $x(t)$ ranging from $\sim\epsilon$ to an order of unity. In the first case considered in Section 2 (i.e., passage from an extensive regime), the number of meshes and hence the time for computation can be enormously reduced (at least by a factor of 10) by continuously modifying the range in which $x(t)$ is computed and the value of Δx, depending on the mean and variance of $x(t)$. This procedure is, however, not applicable in the case of evolution from intrinsically unstable states.

The method has the advantage of being a numerically exact procedure to obtain the entire distribution function $P(x, t)$ in the complete time domain. No other approximation is involved, when reasonable mesh widths are chosen. Also, the generalization to higher dimensions is straightforward.

3.3. System Size Expansion

System size expansion[5] proceeds with the assumption that the stochastic process $x(t)$ could be split into a mean path (deterministic path) $y(t)$ and a corrective remainder $\sqrt{\epsilon}\,\xi(t)$. Then the distribution function for $\xi(t)$ satisfies

$$\pi(\xi, t)\, d\xi = P(x, t)\, dx \tag{6a}$$

with

$$x(t) = y(t) + \sqrt{\epsilon}\,\xi(t) \tag{6b}$$

Substituting the above expressions in Eq. (3) and equating coefficients of the same power of ϵ, we get to the order of ϵ:

$$\frac{d}{dt}\,y = \gamma y - g y^3 = C_1(y) \tag{7}$$

$$\frac{\partial}{\partial t}\,\pi(\xi, t) = (-\gamma + 3gy^2)\,\frac{\partial}{\partial \xi}\left[\xi \pi(\xi, t)\right] + \frac{\partial^2 \pi}{\partial \xi^2} \tag{8}$$

In Eq. (7), $C_1(y)$ is the first jump moment. The solution is given by

$$\pi(\xi, t) = \left[2\pi\sigma^2(t)\right]^{-1/2} \exp\left[-\frac{\xi^2}{2\sigma^2(t)}\right] \tag{9}$$

where $\sigma^2(t)$ is the variance of ξ, obtained as the solution of the following equation:

$$\frac{d}{dt}\,\sigma^2(t) = 2\left[(\gamma - 3gy^2)\sigma^2(t) + 1\right] \tag{10a}$$

Alternatively, σ_x^2 [the variance of $x(t)$] satisfies

$$\frac{d}{dt}\,\sigma_x^2 = 2\left[(\gamma - 3gy^2)\sigma^2 + \epsilon\right] \tag{10b}$$

An explicit solution of $\sigma^2(t)$ is given by

$$\sigma^2(t) = \frac{\sigma_0^2(\gamma y - g y^3)^2}{(\gamma y_0 - g y_0^3)^2} + \left\{ \frac{g(\gamma y - g y^3)^2}{\gamma^4} \left[3 \ln \frac{gy^2/\gamma}{1 - gy^2/\gamma} + \frac{2}{1 - gy^2/\gamma} + \frac{1}{2(1 - gy^2/\gamma)} - \frac{\gamma}{gy^2} \right]_{y_0}^{y(t)} \right\} \tag{11}$$

where the deterministic path $y(t)$ is given by

$$y(t) = \frac{y_0}{\left\{(g/\gamma)y_0^2 + \left[1 - (g/\gamma)y_0^2\right] e^{-2\gamma t}\right\}^{1/2}}, \qquad y_0 = y(0) \tag{12}$$

This method is applicable only in the first case (i.e., $\delta^2 \gg \epsilon$). In the second case of evolution from an intrinsically unstable state (i.e., $\delta^2 \ll \epsilon$) the methods fails. This is because the fluctuations are no longer corrections to the deterministic part, but they completely determine the evolution itself

(i.e., the higher-order terms neglected are as important as the terms retained.)[4]

3.4. Generalized Statistical Linearization Scheme

The basic idea of this method is to replace the original nonlinear Langevin equation [Eq. (1)] by an equivalent linear equation[6–9]:

$$\frac{d}{dt}x = \tilde{\gamma}(t)x(t) + \tilde{C}(t) + \eta(t) \tag{13}$$

Having made such a replacement we cannot hope to reproduce all the non-Gaussian features associated with Eq. (1). However, with the leverage of two arbitrary functions $\tilde{\gamma}(t)$ and $\tilde{C}(t)$ available, we may at least obtain the first two moments of $x(t)$ in some optimal sense. This is achieved by demanding that the ensemble average of the error due to the replacement of Eq. (1) by Eq. (13) be a minimum.[6] This leads to some expressions for the optimal choice of $\tilde{\gamma}(t)$ and $\tilde{C}(t)$. We now derive equations for the evolution of $\langle x \rangle$ and $\langle x^2 \rangle$ starting from the equivalent linear equation [Eq. (13)]. Substituting for $\tilde{\gamma}(t)$ and $\tilde{C}(t)$ we get

$$\frac{d}{dt}\langle x \rangle = \gamma \langle x \rangle - g\langle x^3 \rangle \tag{14}$$

and

$$\frac{d}{dt}\langle x^2 \rangle = 2\left[\gamma \langle x^2 \rangle - g\langle x^4 \rangle + \epsilon\right] \tag{15}$$

[4] It is to be noted that the above discussion pertains to the system size expansion applied to the "master equation" represented by the Fokker–Planck equation itself. Instead, we may consider Eq. (1), without the random force term $\eta(t)$, to be the macroscopic (average) equation corresponding to some microscopic process. The master equation satisfied by the distribution function can be written down in terms of the forward and backward transition rates.[5] Performing system size expansion on this master equation, we get

$$\frac{\partial}{\partial t}\tilde{\pi}(\xi, t) = (-\gamma + 3gy^2)\frac{\partial}{\partial \xi}[\xi \tilde{\pi}(\xi, t)] + \frac{(\gamma y + gy^3)}{2}\frac{\partial^2}{\partial \xi^2}\tilde{\pi}(\xi, t)$$

with

$$\frac{\partial}{\partial t}y = \gamma y - gy^3$$

The variance of x then satisfies

$$\frac{d}{dt}\sigma_x^2 = 2\left[(\gamma - 3gy^3)\sigma_x^2 + \frac{\epsilon}{2}(\gamma y + gy^3)\right]$$

clearly σ_x^2 depends on the initial variance and initial value of y. For small values of ϵ, this equation for $\sigma_x^2(t)$ will tend to Eq. (10b).

It should be pointed out that these equations are identical to the ones we would obtain starting from the original nonlinear Langevin equation itself. It should be mentioned that this result is general and is applicable for arbitrary dimensions and nonlinearities.[8] Since Eq. (13) is linear, its distribution function is Gaussian. Hence the best approximation is obtained by a Gaussian decoupling of the higher-order moments occurring in Eqs. (14) and (15). Obviously, the above equations are applicable only to the first case of diffusion from an extensive regime. Comparing with Eqs. (7) and (10b) obtained via system size expansion, we note that the two are similar in form, if we identify the right-hand side of Eq. (14) as a "renormalized" jump moment (after decoupling $\langle x^3 \rangle$ and $\langle x^4 \rangle$).

The method fails in the second case ($\delta^2 \ll \epsilon$). With an idea of respecting the symmetry of the final distribution function, we choose $P(x,t)$ to be a bimodal Gaussian $G(x,t)$ of the form

$$G(x,t) = C\left\{H(-x)\exp\left[-\frac{(x+x_1)^2}{2\sigma_1^2}\right] + H(x)\exp\left[-\frac{(x-x_1)^2}{2\sigma_1^2}\right]\right\} \tag{16}$$

where

$$C = \left\{(2\pi)^{1/2}\sigma_1\left[1 + \operatorname{erf}\left(x_1/\sqrt{2}\,\sigma_1\right)\right]\right\}^{-1}$$

Here $H(x)$ is the step function. Since this distribution function is symmetric, all odd moments vanish. Hence, starting from Eq. (1), we can write down the evolution equations for the first two nonvanishing moments $\langle x^2 \rangle$ and $\langle x^4 \rangle$. The higher-order moments occurring in these equations are decoupled using Eq. (16). The unknown functions x_1 and σ_1 are then determined self-consistently.

3.5. THE SCALING THEORY

Many authors[2–4,11–15] have studied the problem of evolution from an intrinsically unstable state. The method we discuss is due to Suzuki.[2] The idea is to go over to a transformed variable which approximately satisfies a linear equation. One of the many nonlinear transformations used is

$$\xi = F^{-1}\left[e^{-\gamma t}F(x)\right] \tag{17}$$

with

$$F(x) = \exp\left[\int_{a_0}^{x} dy\, \gamma / C_1(y)\right], \qquad C_1(y) = \gamma y - g y^3 \tag{18}$$

Then the Fokker–Planck equation for $P(\xi, t)$ reads

$$\frac{\partial}{\partial t} P(\xi, t) = \epsilon \frac{\partial}{\partial \xi} \left\{ \frac{C_1(\xi)}{C_1(x)} \frac{\partial}{\partial \xi} \left[\frac{C_1(\xi)}{C_1(x)} P(\xi, t) \right] \right\} \tag{19}$$

In the limit of small ϵ and large t, it is possible to show that

$$\frac{C_1(\xi)}{C_1(x)} \simeq e^{-\gamma t} \tag{20}$$

The resulting FPE can be easily solved. Transforming back to the original variable, we get for $x(0) = 0$,

$$P(x, t) = \frac{F'(x)}{(2\pi\tau)^{1/2} F'(\xi(x))} \exp\left[- \frac{\left(\xi(x) e^{\gamma t}\right)^2}{2\tau} \right] \tag{21}$$

where

$$\tau = e^{2\gamma t}\left[\sigma_0^2 + \frac{\epsilon}{\gamma}(1 - e^{-2\gamma t}) \right]$$

and σ_0^2 is the initial variance. [In the above the prime ($'$) refers to differentiation with respect to the argument of the function.]

As an improvement over the above-mentioned scaling theory, Suzuki has proposed the renormalized perturbation scheme.[17] In the infinite-order limit, this is expected to lead to the exact result. The scaling theory follows in the zeroth-order approximation. In the first-order perturbation theory, he obtains the system size expansion result for the extensive regime. We point out that there is a difference between them, namely, the initial fluctuations are not damped out in this perturbative approach.

4. COMPARATIVE STUDY OF VARIOUS METHODS OF SOLUTION

In this section, we will compare the relative performance of the different methods of solutions considered in the previous sections

4.1. Passage from an Apparently Unstable State

We choose the initial distribution function to be a Gaussian. We set $\delta = 5 \times 10^{-3}$, $\sigma_0^2 = 1 \times 10^{-6}$, $\epsilon = 0.5 \times 10^{-6}$ and $\gamma = g = 1$. We have obtained $P(x, t)$ from the numerical solution of FPE with $\Delta x = 0.005$ and $\Delta t = 0.001$. The non-Gaussian feature of $P(x, t)$ is clearly seen in Fig. 1, which depicts $P(x, t)$ for three typical times. The total computer time taken was about 5 min in the Honeywell Bull DPS8 system.

In the Monte Carlo method of solution, averages were obtained with an ensemble of 4900 tracks. This number of tracks was found to be

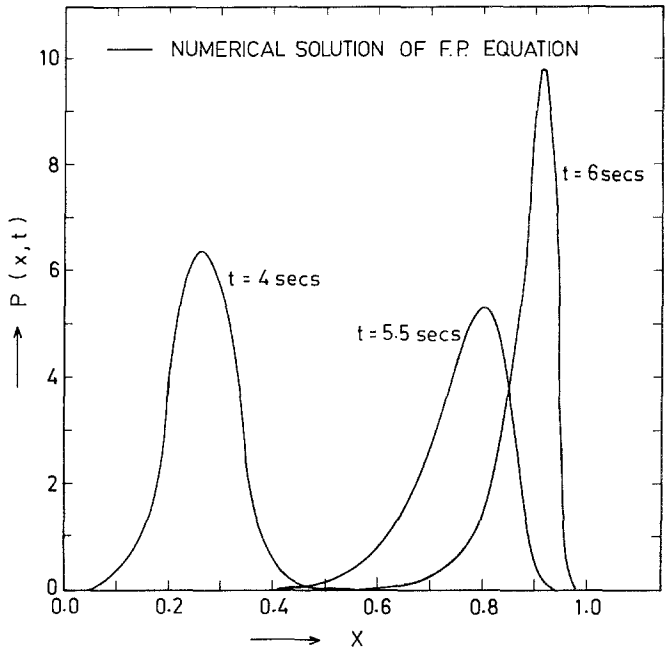

Fig. 1. The distribution function $P(x, t)$ obtained by the numerical solution of the Fokker–Planck equation for the case of evolution from the extensive regime with $\delta = 0.005$; $\sigma_0^2 = 1 \times 10^{-6}$ and $\epsilon = 0.5 \times 10^{-6}$.

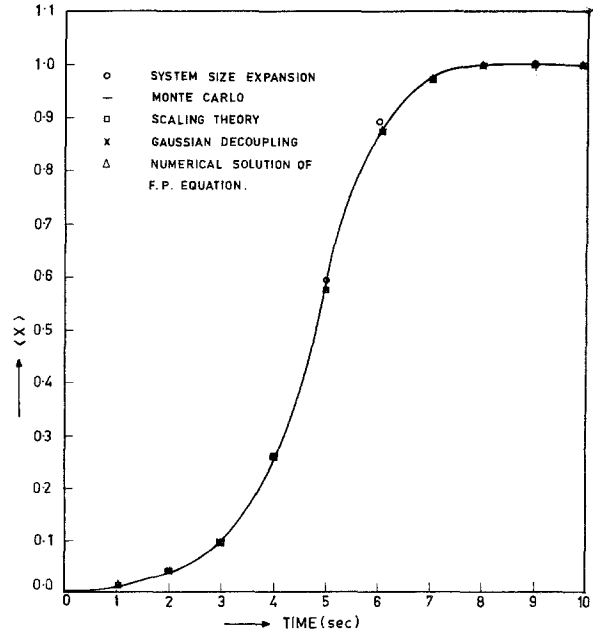

Fig. 2. $\langle x(t) \rangle$ as a function of time, for the problem considered in Fig. 1.

sufficient to obtain a good statistical estimate of the first and second moments. The computer time required was around 50 min.

We shall now compare the first two moments obtained by various methods discussed in Section 3. We shall treat the solution obtained by Monte Carlo technique as the exact solution against which others are compared. We do not present here the results of the scaling theory because it has been demonstrated to be almost equivalent to the system size expansion, for the evolution from the extensive regime under consideration. We just remark in passing that the variance obtained by this method shows deviations at large times as a consequence of the initial fluctuations not being damped out (see also Section 3.5). Figure 2 shows a plot of $\langle x(t) \rangle$ as a function of time. All methods discussed compare well. Figure 3 shows a plot of the variance. All the methods show fluctuation enhancement characteristic of a nonlinear system. The variance obtained from the numerical solution of FPE compares extremely well with that obtained by the Monte Carlo method. Here it should be emphasized that the numerical solution of FPE is several times faster than the Monte Carlo method. The Gaussian decoupling scheme compares well but for a small discrepancy at the intermediate time. This is due to the severe non-Gaussian feature of the

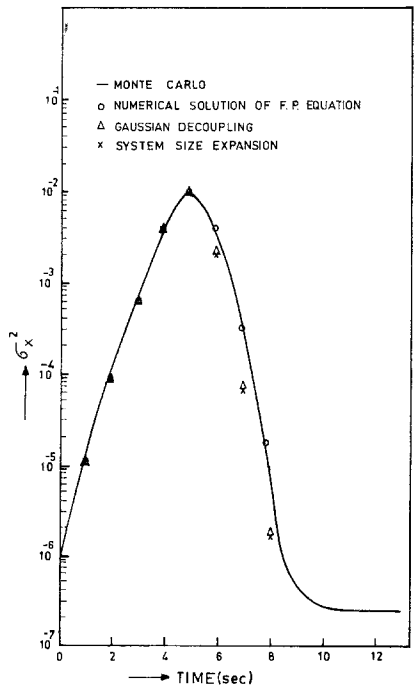

Fig. 3. The variance of $x(t)$ as a function of time for the problem considered in Fig. 1.

actual $P(x, t)$, as can be seen in Fig. 1. The results of system size expansion are as good as the Gaussian decoupling scheme for the small value of ϵ that is chosen. As discussed earlier, the success of these methods is due to the unimodal character of the distribution function.[5]

It is also clear that when the number of variables (N) is large and if we are interested in the first two moments only, Monte Carlo method becomes more attractive. This is due to the fact that the computer time for numerical solution of FPE will increase as N^2 against the linear increase for the Monte Carlo method.

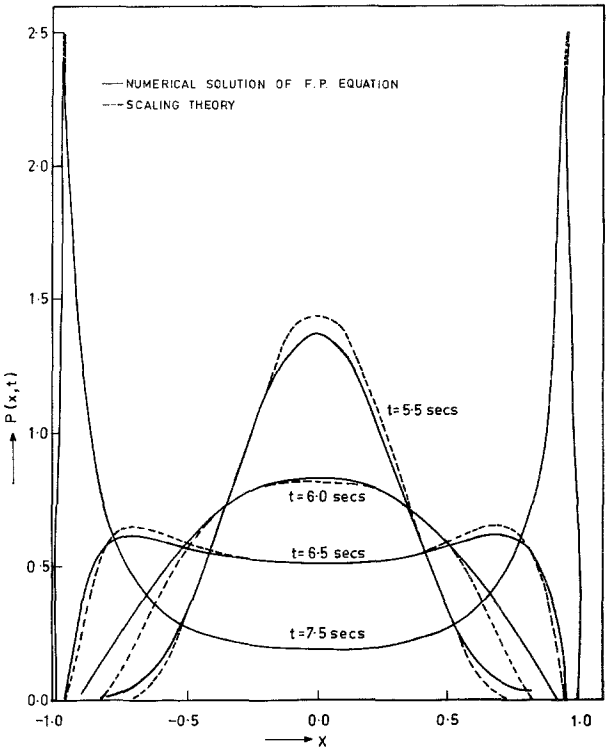

Fig. 4. The distribution function $P(x, t)$ obtained by the numerical solution of the Fokker–Planck equation and from the scaling theory, for the case of evolution from an intrinsically unstable state with $\delta = 0.0$; $\sigma_0^2 = 1 \times 10^{-6}$ and $\epsilon = 0.5 \times 10^{-6}$.

[5] A study of system size expansion applied to the deterministic equation discussed in footnote 4 (Section 3.3), was also made. It was found that the result obtained for $\sigma_x^2(t)$ differed by a factor of 10^{-2}, for a deterministic initial condition (i.e., $\sigma_0^2 = 0$).

4.2. Passage from an Intrinsically Unstable State

When the system relaxes from an intrinsically unstable state, the evolution of fluctuations cannot be obtained by system size expansion, as discussed earlier. The Gaussian decoupling scheme works provided we assume an *a priori* bimodal Gaussian distribution function, for decoupling higher-order moments. Hence we have studied this case using methods other than system size expansion for an initial Gaussian distribution function with $\delta = 0$ and $\sigma_0^2 = 1 \times 10^{-6}$.

Figure 4 shows a plot of $P(x,t)$ obtained by the numerical solution of FPE with $\Delta x = 0.00025$ and $\Delta t = 0.001$. The results of scaling theory are also shown alongwith. The two match well for the time scales considered here. Monte Carlo solution of the nonlinear Langevin equation has been obtained with 4900 tracks. Figure 5 depicts the variance obtained by various methods as a function of time. The numerical solution of FPE compares extremely well with Monte Carlo solution. The scaling theory performs best amongst the approximate methods. As a matter of fact, the agreement is better when the initial condition is deterministic. The bimodal Gaussian decoupling compares reasonably well considering the fact that it is a very simple approximation to the actual non-Gaussian process. The

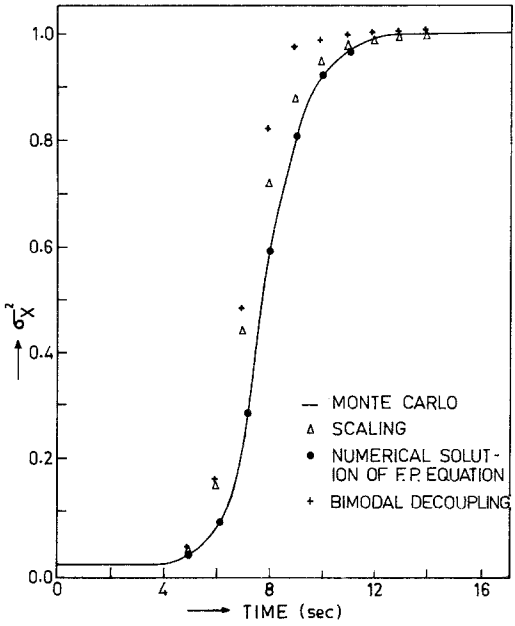

Fig. 5. The variance of $x(t)$ as a function of time for the problem considered in Fig. 4.

deviation is maximum in the intermediate region. For these computations, the Monte Carlo method took 50 min compared to 60 min consumed by the numerical solution of the FPE.

5. CONCLUSIONS

In light of the present investigation the following picture emerges regarding the advantages and limitations of the methods considered. The numerical solution of the Fokker–Planck equation should be very useful in problems where analytical methods fail since it provides the entire distribution function with minimal computer time. On the other hand in situations involving many variables the Monte Carlo method becomes attractive in getting exact values of first and second moments. The Monte Carlo method also seems to be the only one which can be used when the driving random force is not a Gaussian white noise. The Gaussian decoupling scheme, which earlier has been shown to be effective in higher dimensions,[8] after all performs reasonably well even in the case when multiple steady states are allowed. Suzuki's scaling theory performs remarkably well in unraveling the evolution from intrinsically unstable states. System size expansion is comparable to the Gaussian decoupling scheme for the extensive regime.

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